Mahonian statistics for set partitions

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Outline

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Definition

Given a permutation $\pi = \pi_1\pi_2\ldots\pi_n$, a pair (π_i, π_j) is called an inversion of π if $i < j$ and $\pi_i > \pi_j$.

Denote by $\mathsf{inv}(\pi)$ the number of inversions of π.

Example

Let $\pi = 21534$.

Clearly, $(2, 1)$, $(5, 3)$, and $(5, 4)$ are inversions of π.

So $\mathsf{inv}(\pi) = 3$.

Definition

A descent of $\pi = \pi_1\pi_2\ldots\pi_n$ is an index i with $\pi_i > \pi_{i+1}$.

Let $\mathsf{Des}(\pi)$ be the set of descents of π.

Define the major index of π, denoted $\mathsf{maj}(\pi)$, to be

$$\mathsf{maj}(\pi) = \sum_{i\in\mathsf{Des}(\pi)} i.$$

Example

Let $\pi = \pi_1\pi_2\pi_3\pi_4\pi_5 = 21534$.

Since $\pi_1 > \pi_2$ and $\pi_3 > \pi_4$, we have $\mathsf{Des}(21534) = \{1, 3\}$.

Then $\mathsf{maj}(21534) = 1 + 3 = 4$.

MacMahon showed that inv and maj are equidistributed over permutations in 1916.

Theorem (MacMahon, 1916)

For any $n, k \geq 0$, we have

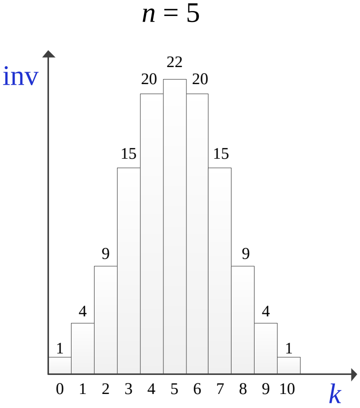
$$|\{\pi \in \mathcal{S}_n : \mathit{inv}(\pi) = k\}| = |\{\pi \in \mathcal{S}_n : \mathit{maj}(\pi) = k\}|,$$

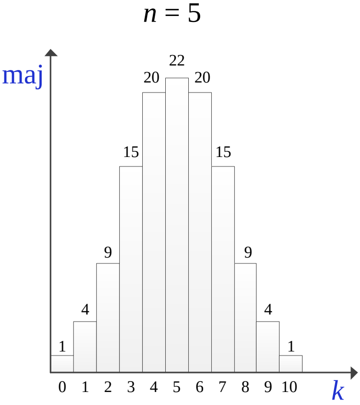
where $\mathcal{S}_n$ is the set of permutations of $\{1, 2, \ldots, n\}$.
Equivalently,

$$\sum_{\pi \in \mathcal{S}_n} q^{\mathit{inv}(\pi)} = \sum_{\pi \in \mathcal{S}_n} q^{\mathit{maj}(\pi)}.$$

- It was not until 1968 that a famous bijective proof was found by Foata.
- Carlitz gave another bijective proof in 1975.

Part 1. Mahonian statistics





$$\sum_{\pi \in \mathcal{S}_n} q^{\mathsf{inv}(\pi)} = \sum_{\pi \in \mathcal{S}_n} q^{\mathsf{maj}(\pi)}$$
$$= q^0 + 4q^1 + 9q^2 + 15q^3 + 20q^4 + 22q^5 + 20q^6 + 15q^7 + 9q^8 + q^{10}.$$

- We use the notation $M = \{1^{k_1}, 2^{k_2}, \dots, m^{k_m}\}$ for the multiset M consisting of k_i copies of i, for all $i \in [m] := \{1, 2, \dots, m\}$.
- Let $\mathcal{S}_M$ be the set of multipermutations of multiset M (the elements of $\mathcal{S}_M$ are called words).
- E.g., let $M = \{1^2, 2^2, 3^2, 4^3\}$, $122443341 \in \mathcal{S}_M$.
- The definitions of inv and maj for words are the same as those for permutations.
- MacMahon's equidistribution theorem holds on $\mathcal{S}_M$.

Theorem (MacMahon, 1916)

For any multiset M, we have

$$\sum_{w \in \mathcal{S}_M} q^{\mathsf{inv}(w)} = \sum_{w \in \mathcal{S}_M} q^{\mathsf{maj}(w)}.$$

In honor of MacMahon, any statistic on permutations or words that is equidistributed with maj and inv is said to be Mahonian.

Table 1: Mahonian statistics on permutations and words.

Name	Reference	Year
inv	Rodriguez	1839
maj	MacMahon	1916
r-maj	Rawlings	1981
maj_d	Kadell	1985
z-index	Zeilberger-Bressoud	1985
den	Denert (for permutations)	1990
	Han (for words)	1994
mak	Foata-Zeilberger (for permutations)	1990
	Clarke-Steingrímsson-Zeng (for words)	1997
mad	Clarke-Steingrímsson-Zeng	1997
stat	Babson-Steingrímsson (for permutations)	2000
	Kitaev-Vajnovszki (for words)	2016

Outline

Part 2. Mahonian statistics for set partitions

Definition

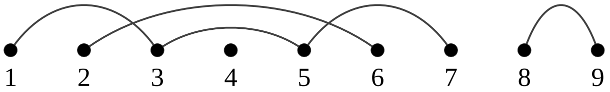
A partition of the set $[n] = \{1, 2, \ldots, n\}$ is a collection of disjoint nonempty subsets (called blocks) of $[n]$, whose union is $[n]$.

In particular, a (perfect) matching of $[n]$ is a partition of $[n]$ in which each block contains exactly two elements.

- E.g., $\{\{1, 3, 5, 7\}, \{2, 6\}, \{4\}, \{8, 9\}\}$ is a partition of $[9]$.
- There are three commonly used representations for set partitions. Each of them has a MacMahon-type result.

Standard representation

- Given a partition of $[n]$, the graph on the vertex set $[n]$ whose edge set consists of the arcs connecting the elements of each block in numerical order is called the standard representation.
- E.g., the standard representation of $\{\{1,3,5,7\},\{2,6\},\{4\},\{8,9\}\}$ has the arc set $\{(1,3),(3,5),(5,7),(2,6),(8,9)\}$.



Part 2. Mahonian statistics for set partitions

Using the standard representation, Chen, Gessel, Yan and Yang (JCTA, 2008) introduced a major index statistic, pmaj, and then obtained a MacMahon-type result for set partitions.

Theorem (Chen-Gessel-Yan-Yang, 2008)

For $n \geqslant 1$, we have

$$\sum_{\pi \in \Pi_n} q^{\mathsf{pmaj}(\pi)} = \sum_{\pi \in \Pi_n} q^{\mathsf{cr}_2(\pi)},$$

where Π_n is the set of all partitions of $[n]$.

W.Y.C. Chen, I. Gessel, C.H. Yan, A.L.B. Yang, A major index for matchings and set partitions, *J. Combin. Theory Ser. A* 115 (2008) 1069-1076.

Block representation

- Given a partition of $[n]$, write it as $B_1/B_2/\cdots/B_m$, where $B_1, B_2, \ldots, B_m$ are the blocks and satisfy the following order property
$$\min B_1 < \min B_2 < \cdots < \min B_m.$$
This representation is called the block representation.
- E.g., the block representation of $\{\{4\},\{2,6\},\{8,9\},\{1,3,5,7\}\}$ is $\{1,3,5,7\}/\{2,6\}/\{4\}/\{8,9\}$.
- From the historical point of view, the set partitions are defined by block representation.

Part 2. Mahonian statistics for set partitions

Using the block representation, Sagan (EJC, 1991) introduced a major index statistic, $\overline{\mathsf{maj}}$, and then obtained a MacMahon-type result for set partitions.

Theorem (Sagan, 1991)

For $n, m \geqslant 1$, we have

$$\sum_{\pi \in \Pi_{n,m}} q^{\overline{\mathsf{maj}}(\pi)} = \sum_{\pi \in \Pi_{n,m}} q^{\overline{\mathsf{inv}}(\pi)},$$

where $\Pi_{n,m}$ is the set of all partitions of $[n]$ with exactly m blocks.

B. Sagan, A maj statistic for set partitions, *European J. Combin.* 12 (1991) 69-79.

Canonical representation (restricted growth functions)

- Given a partition w of $[n]$ with blocks $B_1, B_2, \cdots, B_m$, where

$$\min B_1 < \min B_2 < \cdots < \min B_m,$$

  we write $w = w_1 w_2 \ldots w_n$, where w_i is the block number in which i appears, that is, $i \in B_{w_i}$. This representation is called the canonical representation.

- E.g., consider the set partition $\{\{1,3,5,7\},\{4\},\{2,6\},\{7,8\}\}$.

  (1) Write it as block representation $\{1,3,5,7\}/\{2,6\}/\{4\}/\{8,9\}$.

  (2) Let
  $$w_1 = w_3 = w_5 = w_7 = 1,\ w_2 = w_6 = 2,\ w_4 = 3,\ w_8 = w_9 = 4.$$

  (3) The canonical representation the above set partition is $w = w_1 w_2 \ldots w_9 = 121312144$.

Part 2. Mahonian statistics for set partitions

- Given a word $w = w_1 w_2 \dots w_n$, define the head permutation of it to be the subword consisting of the first occurrences of all letters.
- E.g., the head permutation of 42234321124 is 4231.
- The canonical representation of a set partition is a word whose head permutation is $12 \dots m$.
- E.g., the head permutation of the set partition 121312144 is 1234.
- Those words are called restricted growth functions, since

$$w_i \leqslant \max\{w_1, w_2, \dots, w_{i-1}\} + 1.$$

- So set partitions are in bijection with restricted growth functions (via canonical representation), which was first observed by Hutchinson in 1963.
  Since then, the canonical representation becomes one of the most popular representations in the theory of set partitions.

Using the canonical representation (restricted growth functions), Sagan (EJC, 1991) introduced a major index statistic and then obtained a MacMahon-type result for set partitions.

B. Sagan, A maj statistic for set partitions, *European J. Combin.* 12 (1991) 69-79.

Mahonian representation (see Liu, 2022)

- We consider the following representation for set partitions: given a partition w of $[n]$ with blocks $B_1, B_2, \ldots, B_m$, where

$$\max B_1 < \max B_2 < \cdots < \max B_m,$$

  we write $w = w_1 w_2 \ldots w_n$, where w_i is the block number in which i appears, that is, $i \in B_{w_i}$. We will call this representation the Mahonian representation.
- Note that the only difference between this representation and canonical representation is that here we use the ordering according to the maximal element of the blocks.

- E.g., consider the set partition $\{\{1,3,5,7\},\{4\},\{2,6\},\{7,8\}\}$.
  (1) Write it as $\{4\}/\{2,6\}/\{1,3,5,7\}/\{8,9\}$.

  (2) Let
  $$w_4 = 1,\ w_2 = w_6 = 2,\ w_1 = w_3 = w_5 = w_7 = 3,\ w_8 = w_9 = 4.$$

  (3) The Mahonian representation the above set partition is
  $w = w_1 w_2 \dots w_9 = 323132344$
  (the canonical representation is 121312144).
- Given a word $w = w_1 w_2 \dots w_n$, define the tail permutation of it to be the subword consisting of the last occurrences of all letters.
- The Mahonian representation of a set partition is a word whose tail permutation is $12 \dots m$.
- E.g., the tail permutation of the set partition 323132344 is 1234.

Part 2. Mahonian statistics for set partitions

Name	Reference	Year
inv	Rodriguez	1839
maj	MacMahon	1916
r-maj	Rawlings	1981
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	Kitaev-Vajnovszki (for words)	2016

We prove that the Mahonian statistics in the above table, except for stat, are all equidistributed on set partitions via Mahonian representation.

Part 2. Mahonian statistics for set partitions

Theorem (Liu, 2022)

Using Mahonian representation, we have

$$\sum_{w\in\Pi_{n,m}} q^{\mathsf{inv}(w)} = \sum_{w\in\Pi_{n,m}} q^{\mathsf{maj}(w)} = \sum_{w\in\Pi_{n,m}} q^{\mathsf{maj}_d(w)} = \sum_{w\in\Pi_{n,m}} q^{\mathsf{Z}(w)}$$
$$= \sum_{w\in\Pi_{n,m}} q^{r\text{-}\mathsf{maj}(w)} = \sum_{w\in\Pi_{n,m}} q^{\mathsf{den}(w)} = \sum_{w\in\Pi_{n,m}} q^{\mathsf{mak}(w)} = \sum_{w\in\Pi_{n,m}} q^{\mathsf{mad}(w)},$$

where $\Pi_{n,m}$ is the set of all partitions of $[n]$ with exactly m blocks.

S.-H. Liu, Mahonian and Euler-Mahonian statistics for set partitions, *J. Combin. Theory Ser. A* 192 (2022) 105668.

Part 2. Mahonian statistics for set partitions

- Given a word $w = w_1 w_2 \dots w_n \in \mathcal{S}_M$, recall that a descent of w is an index i with $w_i > w_{i+1}$.
- Define an excedance of w to be an index i with $w_i > i$.
- Let $\mathsf{des}(w)$ and $\mathsf{exc}(w)$ be the numbers of descents and excedances of w, respectively.
- MacMahon showed that exc is equidistributed with des in 1915.
- Any statistic on permutations or words that is equidistributed with des and exc is said to be Eulerian.
- Eulerian statistics and Mahonian statistics are the two most important permutation statistics.
- Skandera introduced an Eulerian statistic mstc in 2001, Carnevale extended it to words in 2017.

Part 2. Mahonian statistics for set partitions

Theorem (Liu, 2022)

Using Mahonian representation, we have

$$\sum_{w\in\Pi_{n,m}} q^{\mathsf{des}(w)} = \sum_{w\in\Pi_{n,m}} q^{\mathsf{exc}(w)} = \sum_{w\in\Pi_{n,m}} q^{\mathsf{mstc}(w)}$$

where $\Pi_{n,m}$ is the set of all partitions of $[n]$ with exactly m blocks.

S.-H. Liu, Mahonian and Euler-Mahonian statistics for set partitions, *J. Combin. Theory Ser. A* 192 (2022) 105668.

Outline

1. Mahonian statistics
2. Mahonian statistics for set partitions
3. Generalizations

- A permutation $\tau = \tau_1\tau_2 \dots \tau_m$ of $[m]$ is said to be consecutive if $\{\tau_1, \tau_2, \dots, \tau_i\}$ forms a set of consecutive numbers for all $i \in [m]$.
- E.g., $\tau = 54362718$ is consecutive, whereas $\tau = 54236718$ is not.
- In particular, the increasing permutation $\tau = 12 \dots m$ is consecutive.
- Let $\mathcal{S}_M^\tau$ be the set of words in $\mathcal{S}_M$ with tail permutation being τ.
- E.g, the tail permutation of 331322112441 is 3241, thus, $331322112441 \in \mathcal{S}_M^{3241}$.
- Clearly, let $\tau = 12 \dots m$, we have that $\mathcal{S}_M^\tau$ is the set of partitions of $[n]$ via Mahonian representation.

Part 3. Generalizations

For the statistics inv, maj, maj_d, and z, we prove the following more general result.

Theorem (Liu, 2022)

Let $M = \{1^{k_1}, 2^{k_2}, \ldots, m^{k_m}\}$ with $k_i \geq 1$ for all $i \in [m]$, and let τ be a consecutive permutation of $[m]$, then

$$\sum_{w \in \mathcal{S}_M^\tau} q^{\mathsf{inv}(w)} = \sum_{w \in \mathcal{S}_M^\tau} q^{\mathsf{maj}(w)} = \sum_{w \in \mathcal{S}_M^\tau} q^{\mathsf{maj}_d(w)} = \sum_{w \in \mathcal{S}_M^\tau} q^{\mathsf{z}(w)}.$$

S.-H. Liu, Mahonian and Euler-Mahonian statistics for set partitions, *J. Combin. Theory Ser. A* 192 (2022) 105668.

Thank you!